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Global Observability of Morse–Smale Vectorfields*

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A procedure for investigating the global observability of a class of vectorfields is proposed. The method derives from given qualitative properties of the flow. It is shown that for Morse–Smale flows, local observability criteria can be tied together, leading to a global theorem.

I. INTRODUCTION

A large class of physical dynamical systems are adequately modeled by smooth vectorfields defined on a state space, viz., a differentiable manifold. In general, the state is an internal variable of the system. Hence, an observer might be prevented from observing the evolution of the state directly. He only gets to see a distorted picture of the state evolution, that is, the image under a mapping from the state space to a finite-dimensional vector space.

The basic question related to this set-up is, whether the combination vectorfield–outputmapping is observable, i.e., whether each output trajectory corresponds to a *unique* state trajectory. This question is fundamental to the engineering problem of state estimation and also to the theoretical problem of minimal realization of input–output relations. In this paper it is shown that for Morse–Smale systems with output, a *global* observability criterion can be obtained.

Quite a number of papers have been published on the problem. All of them are local in character, either by restricting attention to a small neighborhood on the manifold, or by considering the output trajectory only in an infinitesimal time interval, and hence leading to weaker results. In this paper one is interested in observability on the whole manifold. Also, the proof of the derived results is inherently connected with (almost) all the

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information available to the observer—the output trajectory over a *finite time* interval.

We will now discuss the relevant literature very briefly. Lee and Markus [1], using an implicit function theorem, obtain a sufficient condition for local observability at equilibrium points of a differential equation on $\mathbb{R}^n$. Kostyukovskii [2] obtains necessary conditions for observability in terms of the successive time derivatives, y, $\dot{y}$, $\ddot{y}$,..., of the output trajectories. Kumar and Griffith [3] have results in the same spirit but in addition they propose a sufficient condition, which, as it turns out, is quite hard to apply to an actual system. At this point it should be said, as a general remark, that in the context of an observability criterion, it is not recommended to incorporate time derivatives of the output. Indeed, the output is always corrupted with noise, which is strongly amplified under derivation. Therefore, conclusions based on such a procedure are highly unreliable. It is emphasized that this article, taking an alternative approach towards the problem, does not suffer from this drawback. Kou, Tarn and Elliott [4] obtain sufficient conditions for global observability for nonlinear differential equations on $\mathbb{R}^n$. The underlying idea is to assure a diffeomorphic correspondence between the state space and an output space. Again this diffeomorphism is constructed by considering successive derivatives. Their results do not seem to be readily extendable to more general manifolds. Inouyé [5] shows that for polynomial differential equations, observability is equivalent with the one-to-one-ness of a nonlinear mapping into a possibly high-dimensional codomain. The mapping again is a "successive derivative" map. Sussmann [6] has recently shown that a deeper treatment of this idea is fruitful in the case of polynomial systems with inputs; "universal" inputs exist which in a single experiment serve to distinguish all states which could be distinguished by a multiple experiment. Parameter identification problems for linear or polynomial systems are covered by his results. Hermann and Krener [7] have set up a theory of local observability which is in a sense "dual" to the known theory of controllability based on Lie algebras of vectorfields.

Section II is preliminary material. In Section III we will prove two local observability criteria which will be used in Section IV. The main results are in Section IV.

II. Preliminaries

We are concerned with systems Σ described by a smooth vectorfield X defined on a smooth finite-dimensional connected manifold M, together with a smooth output mapping $h: M \to \mathbb{R}^p$. Let $\Phi(t, x_0)$ denote an *integral curve* of X with $\Phi(0, x_0) = x_0$; Φ is unique if it exists and is called the *flow* induced by X. The system Σ is called *observable* herein if there exists a positive time T such that for every pair $(x_0, x_1) \in M \times M$ solutions $\Phi(\cdot, x_0)$, $\Phi(\cdot, x_1)$ exist on $[0, T]$, and $x_0 \neq x_1$ implies $h \circ \Phi(t, x_0) \neq h \circ \Phi(t, x_1)$ for some $t \in [0, T]$. We refer to this as *global* observability to stress the distinguishability of all states. A vector field is called *complete* if $\Phi(t, x)$ is defined on $\mathbb{R} \times M$. If M is compact, all its vectorfields are complete. Henceforth we assume completeness.

For the qualitative theory of vectorfields see [9-11]. We now sketch a few useful concepts. If X is a vectorfield on M, an *orbit* of X is the image $\Phi(\mathbb{R}, x)$ of an integral curve on M. A point p of M is called an *equilibrium point* or a *critical point* if $X(p) = 0$; a *closed orbit* γ of X is an orbit which is compact and on which X does not vanish (it corresponds to a periodic solution with period $\tau(\gamma)$). A *critical element* of X is an orbit which is either closed or a singleton equilibrium point. The set of all orbits is the *phase portrait* of f. To describe asymptotic behavior, we define the *ω-limit* [α-limit] *set of* $p \in M$ as: $\{m \in M$ for which there exists a sequence $\{t_n\colon t_n \to \infty\}$ $[\{t_n\colon t_n \to -\infty\}]$ such that $\Phi(p, t_n) \to m\}$. It is denoted by $\omega(p)$ or $\alpha(p)$, respectively. A point $p \in M$ is a *nonwandering point* for the flow Φ on M if, for all neighborhoods U of p and all $t_0 > 0$, there is a $t > t_0$ such that $U \cap \Phi(t, U)$ is nonempty. A critical point p is called *hyperbolic* if, in local coordinates, the Jacobian of $X(p)$ has no eigenvalue with zero real part. A closed orbit γ is called hyperbolic if for $p \in \gamma$, $n - 1$ of the eigenvalues of the Jacobian of $\Phi(\tau(\gamma), p)$ have modulus $\neq 1$. (One eigenvalue must always be 1.) The local behavior of a vectorfield near a hyperbolic critical element is particularly simple since it is topologically equivalent to that of its linearization. (Two flows Φ and Ψ are *topologically equivalent* on an open set U if there exists a homeomorphism γ of U such that $\Phi(t, \gamma(m)) = \gamma \circ \Psi(t, m)$, $t \in \mathbb{R}$, $m \in U$.) Define the *stable manifold* of a critical element C by $W^s(C) = \{p \in M\colon \omega(p) \subset C\}$; for the unstable manifold $W^u(C)$ replace ω by α. $W^s(C)$ and $W^u(C)$, for hyperbolic C, are immersed submanifolds; C lies in their intersection and their tangent spaces at $p \in C$ span T_pM. If two submanifolds have the property that they either (1) do not intersect or (2) at a point of intersection p their tangent spaces span the tangent space T_pM, we say they *intersect transversally.* A vectorfield X is called *structurally stable* if, with respect to a suitable topology on the set of vectorfields, small smooth perturbations of X are topologically equivalent to X. We now consider a class of vectorfields that is important in what follows and in many other respects. A vectorfield on a manifold M is termed *Morse–Smale* if [11, 12]:

(1) The number of fixed points and periodic orbits is finite, and each is hyperbolic.

(2) All stable and unstable manifolds intersect transversally.

(3) The nonwandering set consists of fixed points and periodic orbits only.

It should be pointed out that the set of critical elements is in the limit sets which in turn are in the nonwandering set. What condition (3) together with (1) says, is that the limit sets are finite in number and are exactly the critical elements.

A basic theorem of Peixoto says that when the dimension of the manifold equals two, then the Morse–Smale systems are structurally stable and open and dense. This theorem fails for higher dimensions, but Morse–Smale systems remain structurally stable.

III. Local Observability

The purpose of this section is to develop two local observability theorems. In the next chapter they are used to establish a global observability criterion. First, a few definitions will be introduced. A pair $(x_1, x_2) \in M \times M$ is *distinguishable before time* T, or, *distinguishable over* $[0, T]$, if the output trajectories corresponding to x_1, x_2, respectively, are different for some $t \in [0, T]$. A pair $(x_1, x_2) \in M \times M$ is *indistinguishable before time* T if it is not distinguishable before time T. A system is *observable near* $x_0 \in M$, or, is *locally observable at* x_0, if there exists an open neighborhood U of x_0 and a time T such that every pair of different points $(x_1, x_2) \in U \times U$ is distinguishable before time T. The local theory of observability is developed on $\mathbb{R}^n$, but can be directly extended to manifolds since the derived criterions are independent of coordinate representations.

III.1. *Local Observability at a Fixed Point of a Vectorfield*

We now discuss the observability of a system with output in the neighborhood of a fixed point x_0 of the vectorfield X defined on $\mathbb{R}^n$. Let U be an open neighborhood of x_0 and consider the map

$$U \to C_n[t_0, t_0 + \varepsilon],$$

$$x \mapsto \Phi(t - t_0, x),$$

where t_0, ε are positive real numbers and $\Phi(t - t_0, x)$ is the unique integral curve of the vector field X, with initial value x at time t_0. Here $C_n[t_0, t_0 + \varepsilon]$ denotes the Banach space of all continuous functions $w(\cdot)$ on $t_0 \leqslant t \leqslant t_0 + \varepsilon$ into $\mathbb{R}^n$ with norm

$$\|w\| = \max_{t_0 \leqslant t \leqslant t_0 + \epsilon} [|w_1(t)| + \cdots + |w_n(t)|],$$

where $|\cdot|$ denotes absolute value. Consider also the map

$$C_n[t_0, t_0+\varepsilon] \to C_r[t_0, t_0+\varepsilon],$$
$$\Phi(t-t_0, x) \mapsto h \circ \Phi(t-t_0, x),$$

where h is the output mapping of the system. The derivative of the composite mapping

$$U \to C_r[t_0, t_0+\varepsilon],$$
$$x \mapsto h \circ \Phi(t-t_0, x)$$

evaluated at x_0, is given by

$$\mathbb{R}^n \to C_r[t_0, t_0+\varepsilon],$$
$$\Delta x_0 \mapsto Dh(\Phi(t-t_0, x_0)) \circ \Psi(t)\, \Delta x_0,$$

where $\Psi(t)$ is the solution of the equation of first variation around the orbit $\Phi(t-t_0, x_0)$,

$$\dot{\Psi}(t) = Df(\Phi(t-t_0, x_0))\, \Psi(t),$$
$$\Psi(t_0) = I.$$

Since x_0 is a critical point, the derivative is given by

$$\Delta x_0 \mapsto He^{A(t-t_0)}\, \Delta x_0,$$

where $A = Df(x_0)$ and $H = Dh(x_0)$.

THEOREM (Local Observability at a Fixed Point of a Vectorfield). *The observed process*

$$\dot{x} = f(x), \qquad y = h(x),$$
$$f\colon \mathbb{R}^n \to \mathbb{R}^n,$$
$$h\colon \mathbb{R}^n \to \mathbb{R}^r,$$

with f and h smooth in a neighborhood of the fixed point x_0, is observable near x_0 over the interval $[t_0, t_0+\varepsilon]$ if the observability matrix

$$[H', A'H', \ldots, A'^{n-1}H']$$

has rank n. ($'$ denotes transpose)

Proof. Follows from an implicit mapping theorem [13] once we have proved that, under the above-mentioned condition, the derivative is injective. Assume there exist Δx_1, Δx_2 such that

$$He^{A(t-t_0)}\,\Delta x_1 = He^{A(t-t_0)}\,\Delta x_2 \qquad \text{on} \quad t_0 \leqslant t \leqslant t_0 + \varepsilon.$$

Then, for some $\Delta x_0 = \Delta x_1 - \Delta x_2 \neq 0$,

$$He^{A(t-t_0)}\,\Delta x_0 \equiv 0 \qquad \text{on} \quad t_0 \leqslant t \leqslant t_0 + \varepsilon.$$

Set $t = t_0$, after repeated differentiation, to obtain

$$H\,\Delta x_0 = 0, \qquad HA\,\Delta x_0 = 0, \ldots, HA^{n-1}\,\Delta x_0 = 0.$$

Thus the n rows of the observability matrix are linearly independent, contrary to the hypothesis of the theorem.

For further reference, we will say that *the rank condition is satisfied at a point* x_0, if the observability matrix, evaluated at x_0, has rank n. Notice that the rank condition is independent of the observation interval.

III.2. *Local Observability at a Point on a Periodic Orbit*

In order to construct a local observability criterion near a periodic orbit, we will initially follow a similar argument as above. Let a periodic orbit $\Phi(t - t_0, x_0)$ with period τ be a solution of the vectorfield. Assume, the observation interval is $[t_0, t_0 + (n-1)\tau]$. Define the mapping

$$U \to C_r[t_0, t_0 + (n-1)\tau],$$

$$x \longmapsto h \circ \Phi(t - t_0, x).$$

As before we evaluate the derivative of this mapping at a point x_0 on the periodic orbit. This derivative is given by

$$R^n \to C_r[t_0, t_0 + (n-1)\tau],$$

$$\Delta x_0 \longmapsto Dh(\Phi(t - t_0, x_0)) \circ D\Phi(t - t_0, x_0)\,\Delta x_0.$$

A criterion for local observability is now provided by the implicit mapping theorem if one is assured of the injectiveness of this derivative. Assume that injectiveness is not satisfied, then there exists a certain Δx_0 such that

$$Dh(\Phi(t - t_0, x_0)) \circ D\Phi(t - t_0, x_0)\,\Delta x_0 \equiv 0, \qquad t_0 \leqslant t \leqslant t_0 + (n-1)\tau.$$

Evaluate this expression at $t = t_0, t_0 + \tau, \ldots, t_0 + (n-1)\tau$, then

$$\begin{aligned} &Dh(x_0)\,\Delta x_0 = 0,\\ &Dh(x_0) \circ D\Phi(\tau, x_0)\,\Delta x_0 = 0,\\ &\qquad\vdots\\ &Dh(x_0) \circ D\Phi((n-1)\tau, x_0)\,\Delta x_0 = 0. \end{aligned}$$

The following theorem is now immediate.

THEOREM (Local Observability at a Point on a Periodic Orbit). *Let $\dot{x} = f(x)$, $y = h(x)$ satisfy the above-mentioned conditions. If x_0 at time t_0 is a point belonging to a periodic orbit, then there exists a neighborhood of x_0 such that all pairs of points in that neighborhood are distinguishable over the closed interval $[t_0, t_0 + (n-1)\tau]$ if the rank condition at x_0 is satisfied, i.e., if*

$$\operatorname{rank}[Dh'(x_0), D\Phi'(\tau, x_0) \circ Dh'(x_0), \ldots, D\Phi'((n-1)\tau, x_0) \circ Dh'(x_0)] = n.$$

Note that the criterion is independent of the initial time of the observation interval. It depends on the period of the periodic orbit.

III.3. *Local Observability in the Neighborhood of a Periodic Orbit.*

The previous theorem is not satisfactory for the purpose we have in mind. We want to construct *a neighborhood of the periodic orbit*—not just a neighborhood of a point on the closed orbit—in which all pairs of different points are distinguishable. A system having this property will be said to be *observable in the neighborhood of a closed orbit.* First we quote a short lemma without proof.

LEMMA. *Let Γ be a closed orbit. Given an ε-tubular neighborhood of Γ and a positive time T, there exists a δ-tubular neighborhood of Γ, such that integral curves starting in the δ-neighborhood do not leave the ε-neighborhood before time T.*

THEOREM (Observability in the Neighborhood of a Closed Orbit). *A differential equation with output defined on $\mathbb{R}^n$ (with necessary regularity conditions), having a closed orbit of period τ, is observable within finite time in the neighborhood of a closed orbit if the rank condition is satisfied at some point on the orbit and if the output trajectory corresponding to the closed orbit—and therefore periodic with period τ—has no smaller period.*

Proof. Assume that the rank condition is satisfied at the point x_0 on the orbit. By the previous theorem there is an open neighborhood U_1 containing

x_0, such that any pair of different points $(x_1, x_2) \in U \times U$ is distinguishable in finite time. Let Φ denote the flow corresponding to the vectorfield on $\mathbb{R}^n$. Since for any $t \in \mathbb{R}$, $\Phi_t: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, it follows that $\bigcup_{-\tau \leqslant t \leqslant 0} \Phi_t(U_1)$ is an open neighborhood of the closed orbit. This neighborhood is set up as an uncountable covering. Since the closed orbit, from now on denoted by C, is compact, there exists a finite subcovering $U = \bigcup_1^k U_i$. It is clear that there exists a closed tubular neighborhood $\bar{S}$ covered by U. Since $\bar{S}$ is compact one can associate a Lebesgue number ε with $\bar{S}$ corresponding to the covering U. In other words, points in $\bar{S}$ which are less than ε apart are covered by U_i for some i. From now on we will restrict attention to points in the open tubular neighborhood $S_{\varepsilon/4}$ with "sectional" radius $\varepsilon/4$. There are two means of distinguishing different points x and y belonging to $S_{\varepsilon/4}$:

(i) There exists an open set U_i for some $1 \leqslant i \leqslant k$ which contains both x and y. Then, by construction, after some time less than τ, $\Phi(t, x)$ and $\Phi(t, y)$ are inside U_1. Hence, x and y are distinguishable in finite time, by local observability at x_0.

(ii) There does not exist an open set U_i for some $1 \leqslant i \leqslant k$, containing both x and y. Hence, $d(x, y) > \varepsilon$.

We will now consider case (ii). It will be shown that there exists a tubular neighborhood S_β of C such that any two points x, y in S_β with $d(x, y) > \varepsilon$ are distinguishable in finite time. The proof relies on the assumption that different points on C are distinguishable in finite time. Assume $x, y \in S_{\varepsilon/4}$ with $d(x, y) > \varepsilon$. Let $d(x, C)$ be the distance of x to the periodic orbit C. Since C is compact, there actually is a point $z \in C$ such that $d(x, z) = d(x, C)$. Similarly, there is a point $w \in C$, such that $d(y, w) = d(y, C)$. By the triangle inequality it follows that $d(z, w) \geqslant \varepsilon - 2 \cdot \varepsilon/4 \geqslant \varepsilon/2$. Hence z and w do not coincide. Now, with any z, w on C there corresponds a time segment less or equal than $\tau/2$ representing the time for z to reach w along the flow (or for w to reach z). Let T be the infimum of all the time segments, corresponding to pairs $(z, w) \in C \times C$ which are $\varepsilon/2$ apart. It is clear that T is nonzero and positive. Consider the output trajectory $h^x \in C_r[0, \tau]$ corresponding to the state trajectory which at $t = 0$ is at $x \in C$. Notice that $h^x(0) = h(x)$ with h the output mapping. A different point $y \in C$ gives a different output curve. Let $O(h^x)$ be an open neighborhood of h^x in the function space $C_r[0, \tau]$. We will now construct open neighborhoods $O(h^x)$ about curves h^x, such that if x and y are separated in time along the flow by a period bounded by T and $\tau/2$ time units, there is always a point t_0 with $0 \leqslant t_0 \leqslant \tau$ such that the functional values at t_0 of the functions belonging to $O(h^x)$, on the one hand, and $O(h^y)$, on the other hand, are disjoint. First, we will construct a neighborhood which has the required property if only a "time shift" T is allowed. Let $x \in C$ and $y \in C$ with y related to x as being T time units away from x along the flow. Let $\delta^{x,T} = \|h^x - h^y\|$. Evaluate $\delta^{x,T}$ for any $x \in C$ and associated $y \in C$. We obtain a set of positive real numbers $\{\delta^{x,T} : x \in C\}$. Each $\delta^{x,T}$ is positive. This follows from the assumption that the output trajectory corresponding to the closed orbit has minimal period τ. Let $\delta^T = \inf_{x \in C}\{\delta^{x,T}\}$. If $\delta^T = 0$, then by the compactness of C there exist two points x and y of C for which $\delta^{x,T} = 0$. This contradicts the assumption of the theorem; hence $\delta^T \neq 0$. The above construction is now repeated for each shift larger than T and less than $\tau/2$. We obtain a set of positive numbers δ^t with $T \leqslant t \leqslant \tau/2$. Let $\delta = \inf_{T \leqslant t \leqslant \tau/2} \delta^t$. Again $\delta \neq 0$ by compactness of both $[T, \tau/2]$ and the closed orbit S.

Consider now for any $x \in C$, a sphere $O(h^x)$ about h^x with radius $\delta/4$ in the metric of the function space. These spheres $O(h^x)$ satisfy the necessary requirements. By the lemma and the continuity of the output mapping, one has a tubular neighborhood S_β of the orbit C such that if $x \in S_\beta$, $x \notin C$, and y is a point on C closest to x, then h^x belongs to the $\delta/4$-sphere of h^y. It is immediate, that any two points in S_β at a distance larger than ε and therefore separated in time by at least T and at the most $\tau/2$ time units are distinguishable over a finite time interval equal to the period of the closed orbit. This is true because the output curves belong to neighborhoods such that the functional values, at some $t_0 \in [0, \tau]$, of functions belonging to different $\delta/4$-spheres, are disjoint. Intersect S_β with $S_{\epsilon/4}$ and pick an α such that S_α is in the intersection. Any two points x, y in S_α are distinguishable in finite time. This ends the proof.

IV. Global Observability for Nonlinear Autonomous Vectorfields

In this section a sufficient condition for global observability is derived for a class of smooth vectorfields defined on compact manifolds. As to the isolation of a particular class, it is remarked that for global observability of a vectorfield with infinitely many critical elements, it is *necessary* that an *infinite* number of conditions are satisfied. For this reason and also because of their nice mathematical structure, it is most reasonable to investigate Morse–Smale vectorfields with respect to their global observability properties.

It is the purpose of this section to show how the assumption of Morse–Smale-ness of the vectorfield enables us to assure global observability in terms of a useful algebraic criterion. The approach to the problem essentially consists in tying together the local results described in the previous section by means of a topological argument.

It is well known [11] that for Morse–Smale vectorfields the closure of the stable manifold W_k corresponding with the critical element k is the union of certain W_l. Let $W_l \leqslant W_k$ if $W_l \subset \mathrm{cl}(W_k)$. Then $\leqslant$ is a partial ordering. If $W_l \leqslant W_k$, $l \neq k$, then $\dim W_l \leqslant \dim W_k$, with a strict inequality, in the absence of periodic orbits.

First, as a lemma, a few remarks will be stated. The lemma is an immediate consequence of the compactness of the manifold and the continuity properties of the flow. In what follows the term "neighborhood" will always stand for "open neighborhood."

LEMMA. *Let N_s be a neighborhood of a critical element s of a Morse–Smale vectorfield. Let W_s be its stable manifold. Let W_s' be a subset of W_s obtained after set-theoretic subtraction of a neighborhood of* $(\mathrm{cl}(W_s)\backslash W_s)$.

Then, given W_s,

(i) *with any such W_s', there corresponds a set P of positive real numbers such that*

(a) $T \in P \Rightarrow t \in P$ *with* $t \geqslant T$,

(b) $\Phi(T, W_s') \subset N_s$, $\forall T \in P$,

(ii) *with any such W_s' and a corresponding $T \in P$, there exists an open set $V_s^T \supset W_s'$ such that $\Phi(T, V_s^T) \subset N_s$.*

(iii) *with each real number T, one can associate an open set V_s^T such that $\Phi(T, V_s^T) \subset N_s$, with V_s^T covering W_s up to a neighborhood of* $(\mathrm{cl}(W_s)\backslash W_s)$. *By taking T large enough, V_s^T can be made to cover W_s up to a neighborhood of* $(\mathrm{cl}(W_s)\backslash W_s)$, *no matter how small this neighborhood is taken.*

Before stating the main theorem, we will prove a slightly more restricted result.

THEOREM. *Given a Morse–Smale system on a compact manifold M with a nonzero number of critical points constituting the limit sets. Given also a smooth output function h into euclidean space $\mathbb{R}^r$. Then the system is globally observable within finite time if:*

(1) *the rank condition is satisfied at the critical points,*

(2) *h separates critical points.*

A priori remarks on the course of the proof. For any critical point x_i, consider a neighborhood $O_{h(x_i)}$ of $h(x_i)$. These neighborhoods are taken to be disjoint. This is possible by condition (2) of the theorem. Consider $h^{-1}(O_{h(x_i)})$ for any x_i, and let T_{x_i} be a neighborhood of x_i in the inverse image. Since the rank condition is satisfied, one can associate a neighborhood with each critical point, such that any two points in the same neighborhood are distinguishable in finite time. Intersect now any such neighborhood with its corresponding neighborhood T_{x_i}. Then with each critical point x_i, there corresponds a new neighborhood N_{x_i}. The neighborhoods N_{x_i} play a basic role in the course of the proof. It is noticed that any two points $x \in N_{x_i}$, $y \in N_{x_j}$ are distinguishable by the rank condition (denoted by RC) if $i = j$, and by mutual seperation through h (denoted by MS), if $i \neq j$.

Loosely speaking, the proof of global observability is carried out by selecting subsets of the manifold such that at some specific time they are "pushed forward" by the flow into some of the neighborhoods N_{x_i}, where they are distinguished at some well-picked time instant, either by RC or MS. This will now be illustrated through a discussion of a particular example. The example is not needed in the development of the general theorem, but apart from its simplicity, it sets part of the ideas and it has the property of giving a hint as to how a general proof should run. Consider the unit sphere $S^2 \subset R^3$ centered at the origin. A flow is defined on S^2 with two critical points, a source in the "north-pole" $x_n = (0, 0, 1)$ and a sink in the "south-pole" $x_s = (0, 0, -1)$. The other orbits of the flow are the "meridian lines." Let h be an output function which assumes different values at x_n and x_s. Let the rank condition be satisfied at both poles. Let N_{x_n} and N_{x_s} be neighborhoods of the critical points as described above. Let V_{x_s} be an open set covering M less P_{x_n}, where $P_{x_n} \subset N_{x_n}$ is a neighborhood of x_n. By the lemma, any pair of points in V_{x_s} is distinguishable at some time T_α by RC at x_s. There remains to be shown how to distinguish V_{x_s} from its complement in M, or, to be sure, from N_{x_n} which contains the complement. This is carried out in two steps. First, by the lemma, corresponding to T_α, there exists a neighborhood V_{x_n} of x_n, contained in N_{x_n}, such that $\Phi(T_\alpha, V_{x_n}) \subset N_{x_n}$. Therefore, V_{x_s} is distinguishable from V_{x_n} by MS at time T_α. Finally, again by the lemma, V_{x_s} is distinguishable from N_{x_n} less V_{x_n} by RC in x_s at some finite time $T_\beta > T_\alpha$. This ends the discussion of the example. It is remarked that in a general proof we have to face the "burden" of saddle points. This adds conceptual and technical difficulty. Indeed, at this point it is clear—following the ideas explained in the example—how to construct a proof of global observability for the case of a Morse–Smale vectorfield containing a finite number of sources and sinks and no saddles. When saddles are present, one might at first consider neighborhoods $N_{x_{sa}}$ around the saddle points. Then one might be tempted to say that, since all points—except for the sources—eventually wind up in the neighborhoods N_x of the sinks and the saddles, the example again contains all the ideas on how to give a proof in the general case. Such a reasoning would indeed show how to distinguish all pairs of points on the manifold if one is willing to accept an *infinitely long observation interval.* Indeed, points belonging to the stable manifold of one saddle $N_{x_{sa}}$ but sitting close to the stable manifold of another saddle take a *long time* before they are trapped in $N_{x_{sa}}$—the closer they are to the stable manifold of the other saddle, the longer it takes, by continuity of the flow. These points have a somewhat similar behavior to points in the neighborhood of the north-pole-source of the example. Therefore, in the distinguishability process of a formal proof, these points should somehow be treated together with the stable manifolds to which they are close to—and not together with the stable manifolds to which they belong to—*in order to reduce the observation interval to finite time.* We will now proceed with a formal proof. The proof goes by induction. It relies heavily on the cellular structure induced on the manifold by the stable manifolds of the Morse–Smale vectorfield.

Proof of the theorem. Let A^{so} and A^{si} denote the set of the sources, resp. the sinks, of the vectorfield. Let $A^{sa,i}$ denote the set of saddles with i-dimensional stable manifold. For each i, let $A_+^{sa,i}$ be the union of $A^{sa,i}$ and the stable manifolds of the elements of $A^{sa,i}$. Also, the notation established supra remains valid.

First step. Each $x_i \in A^{so}$ has been attributed a neighborhood $N_{x_i}^{so}$. By the construction, any two points in $\bigcup_i N_{x_i}^{so}$, the union taken over all sources, are distinguishable at time zero.

Second step. We build an open cover of $A^{so} \cup A_+^{sa,1}$, such that any two points in this cover are distinguishable in finite time. Recall that, for Morse–Smale vectorfields, the closure of the stable manifold of an element of $A^{sa,1}$ consists of elements of A^{so}. Consider now a particular $x_{sa} \in A^{sa,1}$ with its stable manifold $W_{x_{sa}}$ and also the sources x_i $(i = 1,..., l)$ belonging to $A^{so} \cap \mathrm{cl}(W_{x_{sa}})$. By the lemma, there exists an open neighborhood $V_{x_{sa}}$ and a time $T_\alpha^{sa,1}$ such that $V_{x_{sa}}$ has a nonempty intersection with $N_{x_i}^{so}$ (for all x_i belonging to $A^{so} \cap \mathrm{cl}(W_{x_{sa}})$) and $\Phi(T_\alpha^{sa,1}, V_{x_{sa}}) \subset N_{x_{sa}}^{sa,1}$. By the lemma, corresponding to $T_\alpha^{sa,1}$, there exist neighborhoods O_{x_i} of the sources x_i such that for each source $\Phi(T_\alpha^{sa,1}, O_{x_i}) \subset N_{x_i}^{so}$. Finally "connect" $V_{x_{sa}}$ with each O_{x_i} by open sets $C_{x_i} \subset N_{x_i}^{so}$ such that C_{x_i} is bounded away from x_i, and furthermore such that $Y_{x_{sa}}^{sa,1} := \bigcup_{i=1}^{l} (O_{x_i} \cup C_{x_i}) \cup V_{x_{sa}}$ covers $\mathrm{cl}(W_{x_{sa}})$. It is noticed that, again by the lemma, each C_{x_i} can be chosen such that $(\bigcup_{i=1}^{l} C_{x_i}) \cup V_{x_{sa}}$ is pushed inside $N_{x_{sa}}^{sa,1}$ by the flow at time $T_\beta^{sa,1} > T_\alpha^{sa,1}$. We claim that any two points in $Y_{x_{sa}}^{sa,1}$ are distinguishable in finite time.

Indeed:

(i) Any two points in $V_{x_{sa}} \cup (\bigcup_{i=1}^{l} C_{x_i})$ are distinguishable at time $T_\beta^{sa,1}$ by RC at the saddle point x_{sa}.

(ii) Any point $x \in V_{x_{sa}}$ is distinguishable from any point $y \in O_{x_i}$ at time T_α by MS at x_{sa} and x_i, respectively $(i = 1,..., l)$.

(iii) Any point $x \in C_{x_i}$ is distinguishable from any point $y \in C_{x_j}$ at time zero, by RC at x_i if $i = j$ and by MS if $i \neq j$ ($i, j = 1,..., l$).

(iv) Any point $x \in C_{x_i}$ is distinguishable from any point $y \in O_{x_j}$ at time zero by RC or MS ($i, j = 1,..., l$).

The above process attaches to each $x \in A^{\mathrm{sa},1}$ a neighborhood $Y_x^{\mathrm{sa},1}$ and two real numbers $T_\alpha^{\mathrm{sa},1}(x)$, $T_\beta^{\mathrm{sa},1}(x)$. We repeat the process for each saddle belonging to $A^{\mathrm{sa},1}$. By the lemma it is possible to arrange the neighborhoods $Y_x^{\mathrm{sa},1}$ such that $T_\alpha^{\mathrm{sa},1}(x)$ and $T_\beta^{\mathrm{sa},1}(x)$ are independent of $x \in A^{\mathrm{sa},1}$.

Following a similar line of reasoning as above, again using the rank condition and mutual separation at any two elements of $A^{\mathrm{sa},1}$ finally leads to an open cover $Y_+^{\mathrm{sa},1}$ of $A^{\mathrm{so}} \cup A_+^{\mathrm{sa},1}$ such that any two points in $Y_+^{\mathrm{sa},1}$ are distinguishable.

$(k+1)$th *step.* Given $Y_+^{\mathrm{sa},k-1}$, it will be shown how to find $Y_+^{\mathrm{sa},k}$. Actually, the ideas are already contained in the implementation of the second step. Let $x_{\mathrm{sa}} \in A^{\mathrm{sa},k}$. Again consider an open neighborhood $V_{x_{\mathrm{sa}}}$ and a time $T_\alpha^{\mathrm{sa},k}$ such that $\Phi(T_\alpha^{\mathrm{sa},k}, V_{x_{\mathrm{sa}}}) \subset N_{x_{\mathrm{sa}}}^{\mathrm{sa},k}$ and also such that the subset of $W_{x_{\mathrm{sa}}}$ not covered by $V_{x_{\mathrm{sa}}}$ is contained in $Y_+^{\mathrm{sa},k-1}$. This is possible by the lemma and by the property of Morse–Smale systems that $\mathrm{cl}(W_{x_{\mathrm{sa}}})$ consists of stable manifolds of lower dimension. With $T_\alpha^{\mathrm{sa},k}$, there corresponds an open set $O_{x_{\mathrm{sa}}}$ covering $\mathrm{cl}(W_{x_{\mathrm{sa}}})$ less $W_{x_{\mathrm{sa}}}$ such that $O_{x_{\mathrm{sa}}} \subset Y_+^{\mathrm{sa},k-1}$ and $\Phi(T_\alpha^{\mathrm{sa},k}, O_{x_{\mathrm{sa}}}) \subset Y_+^{\mathrm{sa},k-1}$. Finally "connect" $O_{x_{\mathrm{sa}}}$ with $V_{x_{\mathrm{sa}}}$ by an open set $O_{x_{\mathrm{sa}}}$ with $O_{x_{\mathrm{sa}}} \subset Y_+^{\mathrm{sa},k-1}$ bounded away from $\mathrm{cl}(W_{x_{\mathrm{sa}}})$ less $W_{x_{\mathrm{sa}}}$ and such that it covers the not yet covered part of $W_{x_{\mathrm{sa}}}$. Executing this for each $x_{\mathrm{sa}} \in A^{\mathrm{sa},k}$, after arranging the relevant neighborhoods such that $T_\alpha^{\mathrm{sa},k-1}(x_{\mathrm{sa}})$, $T_\beta^{\mathrm{sa},k-1}(x_{\mathrm{sa}})$ are independent of x_{sa}, one ends up with an open set $Y_+^{\mathrm{sa},k}$.

In this way one has built a sequence of open sets $\{Y_+^{\mathrm{sa},l}\}$ ($l = 1,..., n$) such that $Y_+^{\mathrm{sa},l} \supset A^{\mathrm{so}} \cup \cdots \cup A_+^{\mathrm{sa},l}$. Any two points in $Y_+^{\mathrm{sa},l}$ are distinguishable. The set $Y_+^{\mathrm{sa},n}$ covers each stable manifold of the Morse–Smale vectorfield, and therefore M, since the stable manifolds provide a cellular decomposition of M. This ends the proof.

We now state the main theorem for global observability.

THEOREM. *Given a Morse–Smale system on a compact manifold with a nonzero number of critical elements. Given also a smooth output function h into euclidean space $\mathbb{R}^r$. Then the system is globally observable if*

(1) *the rank condition is satisfied at the critical elements,*

(2) *h separates critical points,*

(3) *the images of the periodic orbits under h are different and every output trajectory corresponding to a closed orbit has minimal period, equal to the period of the closed orbit.*

Proof. For any critical point x_i, take a neighborhood $O_{h(x_i)}$ of $h(x_i)$. If $i \neq j$, then $O_{h(x_i)}$ is chosen disjoint from $O_{h(x_j)}$. Let s_k and s_l $(k \neq l)$ be any two critical elements. Take neighborhoods $V_{h(s_k)}$ and $V_{h(s_l)}$ of the sets $h(s_k)$ and $h(s_l)$, respectively, such that

(i) if s_i is a critical point, then $V_{h(s_i)} \subset O_{h(s_i)}$ $(i = k, l)$,

(ii) $V_{h(s_k)}$ does not cover $h(s_l)$, or $V_{h(s_l)}$ does not cover $h(s_k)$.

Repeat this construction for each couple (s_k, s_l). In this way, with each s_i there corresponds a finite number of open sets $V_{h(s_i)}$. Let $W_{h(s_i)}$ be a neighborhood of s_i in the intersection of the sets $V_{h(s_i)}$ corresponding with s_i. Let the topology of the manifold be metrized by a metric d. For any s_i, consider $h^{-1}(W_{h(s_i)})$, and take an ε-tubular neighborhood $T_{s_i} \subset h^{-1}(W_{h(s_i)})$ if s_i is a periodic orbit and an ε-ball $T_{s_i} \subset h^{-1}(W_{h(s_i)})$ if s_i is a critical point. Since there are only a finite number of critical elements, these ε-neighborhoods can be chosen to be disjoint. Each closed orbit has finite period. Let τ be the largest period. As shown in the previous section, one constructs a δ-neighborhood of a closed orbit or a fixed point, such that if $x \in \delta$-neighborhood and is not on the periodic orbit or is not a critical point, there exists a point y with y on the closed orbit or y fixed, such that $d(\Phi_t(x), \Phi_t(y)) < \varepsilon$ on a time interval with length τ. It is clear that, when all $W_{h(s_i)}$ are taken small enough, then because of continuity of the flow and the output mapping, any two points x, y on the manifold belonging to different δ-neighborhoods of critical elements give rise to different output trajectories and are therefore distinguishable.

We will now prove the global observability of all points on the manifold under the stated assumptions. The local theorems of the previous section associate to every critical element a neighborhood such that any two points in the same neighborhood are distinguishable in finite time. Intersect each neighborhood (they are assumed to be disjoint) of a critical element supplied by the previous section with the δ-neighborhood constructed above. Then with each critical element s, there corresponds a neighborhood N_s. From here on, the argument runs parallel to the argument in the preceding theorem. This ends the proof of the theorem.

V. Concluding Remark

Thus far, the manifolds under consideration were assumed to be compact. The results can be extended to non compact manifolds under additional hypotheses on the vectorfield. Let M be a manifold, X a smooth vectorfield on M such that there exists only a finite number of equilibrium points or periodic orbits, and furthermore assume that all the state trajectories approach these critical elements as $t \to \infty$. Then the theory of the previous section leads to sufficient conditions for observability on M.

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